DEPARTMENT OF MATHEMATICS

Math 3306
(Mathematical Logic)

4th Tutorial Problems

1. If we take ψ to be $\neg\phi$, then $(\phi \vee \psi)$ is a substitution instance of the tautology $(p_0 \vee \neg p_0)$, and so certainly $(\phi \vee \psi)$ is valid. But we can find ϕ such that neither ϕ nor $\neg\phi$ are valid. For example, let ϕ be $P(C_0)$, so $\neg\phi$ is $\neg P(C_0)$. Now ϕ is not valid, for if we take any $\mathfrak{A} = \langle A, R, \ldots, a\rangle$ where $R(a)$ is false, then not$(\mathfrak{A} \models \phi)$ and so ϕ is not valid. And if we take $\mathfrak{A}_1 = \langle A_1, R_1, \ldots, a_1\rangle$ where $R_1(a_1)$ is true, then not$(\mathfrak{A}_1 \models \psi)$ and so ψ is not valid.

Take the ϕ and ψ as above, and let Σ be the empty collection. Then $\Sigma \vdash (\phi \vee \psi)$. However, not$(\Sigma \vdash \phi)$ and not$(\Sigma \vdash \psi)$. For by the Soundness Theorem, if $\mathfrak{A} \models \Sigma$ and $\Sigma \vdash \phi$ then $\mathfrak{A} \models \phi$. So if we can find $\mathfrak{A}$ with $\mathfrak{A} \models \Sigma$ yet not$(\mathfrak{A} \models \phi)$, then not$(\Sigma \vdash \phi)$. But the $\mathfrak{A}$ above is suitable. And similarly from $\mathfrak{A}_1$ above, not$(\Sigma \vdash \psi)$.

2. Take any $\mathfrak{A} = \langle A, \ldots\rangle$ and any assignment $\boldsymbol{a}$. let τ be the given formula. Since τ is an implication, to show $\mathbf{Val}(\tau, \mathfrak{A}, \boldsymbol{a}) = T$ it suffices to suppose $\mathbf{Val}((\exists v_i \psi \implies \phi), \mathfrak{A}, \boldsymbol{a}) = T$ and show that $\mathbf{Val}(\forall v_i(\psi \implies \phi), \mathfrak{A}, \boldsymbol{a}) = T$. So suppose $\mathbf{Val}((\exists v_i \psi \implies \phi), \mathfrak{A}, \boldsymbol{a}) = T$, and for a contradiction suppose $\mathbf{Val}(\forall v_i(\psi \implies \phi), \mathfrak{A}, \boldsymbol{a}) = F$. Now, by definition, $\mathbf{Val}(\forall v_i(\psi \implies \phi), \mathfrak{A}, \boldsymbol{a}) = T$ iff for all $x \in A$, $\mathbf{Val}((\psi \implies \phi), \mathfrak{A}, \boldsymbol{a}(i|x)) = T$. So our assumption means there is $x \in A$ with $\mathbf{Val}((\psi \implies \phi), \mathfrak{A}, \boldsymbol{a}(i|x)) = F$. So for this x, $\mathbf{Val}(\psi, \mathfrak{A}, \boldsymbol{a}(i|x)) = T$ yet $\mathbf{Val}(\phi, \mathfrak{A}, \boldsymbol{a}(i|x)) = F$. Thus $\mathbf{Val}(\exists v_i \psi, \mathfrak{A}, \boldsymbol{a}) = T$, yet $\mathbf{Val}(\phi, \mathfrak{A}, \boldsymbol{a}(i|x)) = F$. Now v_i is not free in ϕ, and $\boldsymbol{a}(i|x)$ agrees with $\boldsymbol{a}$ except on the value given to v_i, so $\mathbf{Val}(\phi, \mathfrak{A}, \boldsymbol{a}) = \mathbf{Val}(\phi, \mathfrak{A}, \boldsymbol{a}(i|x)) = F$. So we have $\mathbf{Val}(\exists v_i \psi, \mathfrak{A}, \boldsymbol{a}) = T$ yet $\mathbf{Val}(\phi, \mathfrak{A}, \boldsymbol{a}) = F$, and hence $\mathbf{Val}((\exists v_i \psi \implies \phi), \mathfrak{A}, \boldsymbol{a}) = F$, contadicting that $\mathbf{Val}((\exists v_i \psi \implies \phi), \mathfrak{A}, \boldsymbol{a}) = T$.

3(a). $\{\forall v \forall w\} \vdash \forall w \forall v \phi$ – see Sheet 8, Q6(a). And note in this proof, Gen is used only on v and w, neither of which are free in $\forall v \forall w \phi$. So, by the Deduction Theorem, $\vdash (\forall v \forall w \phi \implies \forall w \forall v \phi)$.

3(b).

(1) $\{\phi \implies \forall v \psi\} \vdash (\phi \implies \forall v \psi)$

(2) $\{\phi \implies \forall v \psi\} \vdash (\forall v \implies \psi)$ by A2

(3) $\{\phi \implies \forall v \psi\} \vdash ((\phi \implies \forall v \psi) \implies ((\forall v \psi \implies \psi) \implies (\phi \implies \psi)))$ by A1, tautology $((p_0 \implies p_1) \implies ((p_1 \implies p_2) \implies (p_0 \implies p_2)))$

(4) $\{\phi \implies \forall v \psi\} \vdash (\phi \implies \psi)$ (1), (2), (3) by MP twice

(5) $\{\phi \implies \forall v \psi\} \vdash \forall v(\phi \implies \psi)$ (4) by Gen

(6) $\vdash ((\phi \implies \forall \psi) \implies \forall v(\phi \implies \psi))$ (5) by Deduction theorem, noting we used Gen only on v and v is not free in $(\phi \implies \forall v \psi)$.

3(c).

(1) $\{\forall v(\phi \ \& \ \psi)\} \vdash \forall v(\phi \ \& \ \psi)$

(2) $\{\forall v(\phi \ \& \ \psi)\} \vdash (\forall v(\phi \ \& \ \psi) \implies (\phi \ \& \ \psi))$ by A2

(3) $\{\forall v(\phi \ \& \ \psi)\} \vdash (\phi \ \& \ \psi)$ (1), (2) by MP

(4) $\{\forall v(\phi \ \& \ \psi)\} \vdash ((\phi \ \& \ \psi) \implies \phi)$ by A1, tautology $((p_0 \ \& \ p_1) \implies p_0)$

(5) $\{\forall v(\phi \ \& \ \psi)\} \vdash \phi$ (3), (4) by MP

(6) $\{\forall v(\phi \ \& \ \psi)\} \vdash \forall v \phi$ (5) by Gen

(7) $\{\forall v(\phi \ \& \ \psi)\} \vdash \forall v \psi$ similarly

(8) $\{\forall v(\phi \ \& \ \psi)\} \vdash (\forall v \phi \implies (\forall v \psi \implies (\forall v \phi \ \& \ \forall v \psi)))$
by tautology $(p_0 \implies (p_1 \implies (p_0 \ \& \ p_1)))$

(9) $\{\forall v(\phi \ \& \ \psi)\} \vdash (\forall v \phi \ \& \ \forall v \psi)$ (6), (7), (8) by MP twice

(10) $\vdash (\forall v(\phi \ \& \ \psi) \implies (\forall v \phi \ \& \ \forall v \psi))$ (9) by Deduction theorem, noting we used Gen only on v, and v is not free in $\forall v(\phi \ \& \ \psi)$.

4(i).

(1) $\Sigma \vdash \neg\sigma$ given

(2) $\Sigma \vdash \forall v \neg\sigma$ (1) by Gen

(3) $\Sigma \vdash (\exists v \sigma \iff \neg\forall v \neg\sigma)$ by A4

(4) $\Sigma \vdash (\forall v \neg\sigma \implies ((\exists v \sigma \iff \neg\forall v \neg\sigma) \implies \neg\exists v \sigma)))$
by tautology $(p_0 \implies ((p_1 \iff \neg p_0) \implies \neg p_1))$

(5) $\Sigma \vdash \neg\exists v \sigma$ (2), (3), (4) by MP twice

(1) $\Sigma \vdash \neg\exists v \sigma$ given

(2) $\Sigma \vdash (\exists v \sigma \iff \neg\forall v \neg\sigma)$ by A4

(3) $\Sigma \vdash (\neg\exists v \sigma \implies ((\exists v \sigma \iff \neg\forall v \neg\sigma) \implies \forall v \neg\sigma))$
by tautology $(\neg p_0 \implies ((p_0 \iff \neg p_1) \implies p_1))$

(4) $\Sigma \vdash \forall v \neg\sigma$ (1), (2), (3) by MP twice

(5) $\Sigma \vdash (\forall v \neg\sigma \implies \neg\sigma)$ by A2

(6) $\Sigma \vdash \neg\sigma$ (4), (5) by MP.

4(ii)(a).

(1) $\{\neg\exists w \exists v \phi\} \vdash \neg\exists w \exists v \phi$

(2) $\{\neg\exists w \exists v \phi\} \vdash \neg\exists v \phi$ by (i), where σ is $\exists v \phi$

(3) $\{\neg\exists w \exists v \phi\} \vdash \neg\phi$ by (i) again

(4) $\{\neg\exists w \exists v \phi\} \vdash \neg\exists w \phi$ by first part of (i)

(5) $\{\neg\exists w \exists v \phi\} \vdash \neg\exists v \exists w \phi$ by (i), where σ is $\exists w \phi$

(6) $\{\exists v \exists w \phi\} \vdash \exists w \exists v \phi$ (5) by Contraposition – note in the proof (taking into account (i)) we shall have used Gen only on v and w, neither of which are free in $\exists w \exists v \phi$ or $\exists v \exists w \phi$, so Contraposition does apply.

4(ii)(b) Put $\Sigma = \{\forall v(\phi \implies \psi), \neg\exists v\psi\}$.

(1) $\Sigma \vdash \forall v(\phi \implies \psi)$

(2) $\Sigma \vdash (\phi \implies \psi)$ — (1), A2 and MP (see lecture notes)

(3) $\Sigma \vdash \neg\exists v\psi$

(4) $\Sigma \vdash \neg\psi$ — (3), by second part of 4(i)

(5) $\Sigma \vdash (\neg\psi \implies ((\phi \implies \psi) \implies \neg\phi))$ — by A1, tautology

$(\neg p_0 \implies ((p_1 \implies p_0) \implies \neg p_1))$

(6) $\Sigma \vdash \neg\phi$ — (2), (4), (5) by MP

(7) $\Sigma \vdash \neg\exists v\phi$ — (6), by first part of 4(i)

(8) $\{\forall v(\phi \implies \psi), \exists v\phi\} \vdash \exists v\psi$ — (7) by Contraposition

(Note in the above proof we would have used Gen on v, but v is not free in $\exists v\phi$ or $\exists v\psi$.)

(9) $\{\forall v(\phi \implies \psi)\} \vdash (\exists v\phi \implies \exists v\psi)$ — (8) by Deduction theorem

(We have used Gen only on v, which is not free in $\exists v\phi$.)

(10) $\vdash (\forall v(\phi \implies \psi) \implies (\exists v\phi \implies \exists v\psi))$ — (9) by Deduction theorem

(Again, v is not free in $\forall v(\phi \implies \psi)$)