1 Suppose $\underline{A} \models \phi$. Then for any assignment $\underset{\sim}{a}$ from A, Val(ϕ, $\underline{A}$, $\underset{\sim}{a}$) = T. So take any assignment $\underset{\sim}{a}'$. In particular, for every $x \in A$, Val(ϕ, $\underline{A}$, $\underset{\sim}{a}'(i|x)$) = T, as these $\underset{\sim}{a}'(i|x)$ are among the assignments $\underset{\sim}{a}$. Thus Val($\forall v_i\, \phi$, $\underline{A}$, $\underset{\sim}{a}'$) = T. Since this is true for any $\underset{\sim}{a}'$, we have $\underline{A} \models \forall v_i\, \phi$.

Suppose $\underline{A} \models \forall v_i\, \phi$ so for any assignment $\underset{\sim}{a}$ from A, Val($\forall v_i\, \phi$, $\underline{A}$, $\underset{\sim}{a}$) = T and so for all $x \in A$, Val(ϕ, $\underline{A}$, $\underset{\sim}{a}(i|x)$) = T. In particular, take $x = a_i$. Then $\underset{\sim}{a}(i|x) = \underset{\sim}{a}$, so Val($\phi$, $\underline{A}$, $\underset{\sim}{a}$) = T. Hence for all $\underset{\sim}{a}$, Val(ϕ, $\underline{A}$, $\underset{\sim}{a}$) = T, and so we have $\underline{A} \models \phi$.

2(a) $(((p_0 \Rightarrow p_1) \Rightarrow p_2) \Rightarrow (p_0 \Rightarrow (p_1 \Rightarrow p_2)))$ is a tautology of propositional calculus (check its truth table !) and the formula is a substitution instance of this, with p_0 replaced by $\forall v_0\, P(v_0, v_0)$, p_1 replaced by $\forall v_1\, P_1(v_1, v_1)$, and p_2 replaced by $P_2(v_2, v_2)$. Hence the formula is a group A1 axiom.

(b) Not an axiom.

(c) Let ϕ be the formula $P(v_0, v_0)$, and let t be the term v_0. Then $\phi(v_0|t)$ is $P(v_0, v_0)$, that is, just ϕ back again, and t is substitutable for v_0 in ϕ. So the formula is $(\forall v_0\, \phi \Rightarrow \phi(v_0|t))$, where t is substitutable for v_0 in ϕ. Hence a group A2 axiom.

(d) Not an axiom.

(e) Let ϕ be $\exists v_1\, P(v_0, v_1)$ and let t be the term v_1. Then certainly $\phi(v_0|t)$ is $\exists v_1\, P(v_1, v_1)$, so the formula is of the form $(\forall v_0\, \phi \Rightarrow \phi(v_0|t))$. However t is *not* substitutable for v_0 in ϕ, since the v_1 which replaces the free v_0 is bound by the $\exists v_1$ quantifier. Hence not an axiom.

(f) Let ϕ be $P(c_0, c_1)$ and let ψ be $P(c_0, v_1)$. The formula is then $(\forall\ v_1\, (\phi \Rightarrow \psi) \Rightarrow (\phi \Rightarrow \forall v_1\, \psi)$ and v_1 is not free in ϕ. Hence a group A3 axiom.

3(a) Take any $\underline{A} = \langle A, R, \ \rangle$ and any assignment $\underset{\sim}{a}$ from A. Since ϕ is an implication $(\forall v_1\, P(v_0, v_1) \Rightarrow P(v_0, v_0))$, to show that Val($\phi$, $\underline{A}$, $\underset{\sim}{a}$) = T, we need only check the case where Val($\forall v_1\, P(v_0, v_1)$, $\underline{A}$, $\underset{\sim}{a}$) = T and show that Val($P(v_0, v_0)$, $\underline{A}$, $\underset{\sim}{a}$) = T. So suppose that Val($\forall v_1\, P(v_0, v_1)$, $\underline{A}$, $\underset{\sim}{a}$) = T ; thus for all $x \in A$, Val($P(v_0, v_1)$, $\underline{A}$, $\underset{\sim}{a}(1|x)$) = T. In particular, with $x = a_0$, Val($P(v_0, v_1)$, $\underline{A}$, $\underset{\sim}{a}(1|a_0)$) = T , and thus $R(a_0, a_0)$ holds (since $\underset{\sim}{a}(1|a_0) = \langle a_0, a_0, a_2, \dots \rangle$). But this means that Val($P(v_0, v_0)$, $\underline{A}$, $\underset{\sim}{a}$) = T , as required.

(b) Take any $\underline{A} = \langle A, R, \ldots \rangle$ and assignment a from A. Note that $\phi(v_0|t)$ is the implication $(\forall v_1\ P(t, v_1) \Rightarrow P(t, t))$ so, as in (a) above, we suppose that $Val(\forall v_1\ P(t, v_1), \underline{A}, a) = T$ and aim to show that $Val(P(t, t), \underline{A}, a) = T$. Thus for all $x \in A$, $Val(P(t, v_1), \underline{A}, a(1|x)) = T$, that is, we have that $R(t[a(1|x)], v_1[a(1|x)])$ is true. Now $v_1[a(1|x)] = x$.

Also recall that t is substitutable for v_0 in ϕ. If v_1 occurred in t, then it (v_1) would get "captured" when t was substituted for v_0 in the part "$\forall v_1\ P(v_0, v_1)$". Hence v_1 does not occur in t. This means that the value assigned to v_1 is not used in evaluating $t[a(1|x)]$ or $t[a]$, and since $a(1|x)$ and a agree except at v_1, we have $t[a(1|x)] = t[a]$. Thus for all $x \in A$, $R(t[a], x)$ is true. In particular, when $x = t[a] \in A$, $R(t[a], t[a])$ is true. But this means that $Val(P(t, t), \underline{A}, a) = T$.

Thus $Val(\phi(v_0|t), \underline{A}, a) = T$, and so $\phi(v_0|t)$ is valid.

(c) We need just one $\underline{A} = \langle A, R, \ldots, f, \ldots \rangle$ and one assignment a for which $Val(\phi(v_0|F_0(v_0)), \underline{A}, a) = F$. Take for example $A = \{a, b\}$, $R = \{(a, b), (b, a)\}$, $f(a) = b$, $f(b) = a$, $a = \langle b, b, b, \ldots \rangle$. Then for all $x \in A$, $R(f(x), x)$ is true, and hence $Val(\forall v_1\ P(F_0(v_1), v_1), \underline{A}, a) = T$. But $R(f(b), f(b))$ is false, and so $Val(P(F_0(v_1), F_0(v_1)), \underline{A}, a) = F$. Hence
$Val((\forall v_1\ P(F_0(v_1), v_1) \Rightarrow P(F_0(v_1), F_0(v_1))), \underline{A}, a) = F$, that is,
$Val(\phi(v_0|F_0(v_1)), \underline{A}, a) = F$. So $\phi(v_0|F_0(v_0))$ is not valid.

4 Suppose that $\Sigma \vdash (\phi \,\&\, \psi)$. Then:

(1) $\Sigma \vdash (\phi \,\&\, \psi)$ by assumption;

(2) $\Sigma \vdash ((\phi \,\&\, \psi) \Rightarrow \phi)$ from tautology $((p_0 \,\&\, p_1) \Rightarrow p_0)$;

(3) $\Sigma \vdash \phi$ by 1 and 2, and Modus Ponens.

Similarly $\Sigma \vdash \psi$, using the tautology $((p_0 \,\&\, p_1) \Rightarrow p_1)$.

Suppose that $\Sigma \vdash \phi$ and $\Sigma \vdash \psi$. Then:

(1) $\Sigma \vdash \phi$ by assumption.

(2) $\Sigma \vdash \psi$ by assumption.

(3) $\Sigma \vdash (\phi \Rightarrow (\psi \Rightarrow (\phi \,\&\, \psi)))$ from tautology $(p_0 \Rightarrow (p_1 \Rightarrow (p_0 \,\&\, p_1)))$.

(4) $\Sigma \vdash (\psi \Rightarrow (\phi \,\&\, \psi))$ by 1 and 3, and Modus Ponens.

(5) $\Sigma \Rightarrow (\phi \,\&\, \psi)$ by 2 and 4, and Modus Ponens.

Similarly $\Sigma \vdash (\phi \vee \psi)$, using the tautology $(p_1 \Rightarrow (p_0 \vee p_1))$.

5 Here $\Sigma = \{\forall v_0 \forall v_1 \phi\}$. Each step of the proofs is given

(1) $\forall v_0 \forall v_1 \phi$ is a formula in Σ;

(2) $(\forall v_0 \forall v_1 \phi \Rightarrow \forall v_1 \phi)$ is a formula in A2, since $(\forall v_1 \phi)(v_0|v_0)$ is just $\forall v_1 \phi$, and v_0 is substitutable for v_0 in $\forall v_1 \phi$;

(3) $\forall v_1 \phi$ by (1) and (2), and Modus Ponens.

(a) We continue the proof from (1) - (3) above

(4) $(\forall v_1 \phi \Rightarrow \phi)$ is a formula in A2, since $\phi(v_0|v_0)$ is ϕ, and v_0 is substitutable for v_0 in ϕ;

(5) ϕ by (3) and (4), and Modus Ponens;

(6) $\forall v_0 \phi$ by (5), and Generalisation;

(7) $\forall v_1 \forall v_0 \phi$ by (6), and Generalisation.

Hence, formulae (1) - (7) make up the required proof.

(b) We again continue from (1) - (3) above

(8) $(\forall v_1 \phi \Rightarrow \phi(v_1|v_0))$ is a formula in A2, provided that v_0 is substitutable for v_1 in ϕ;

(9) $\phi(v_1|v_0)$ by (3) and (8), and Modus Ponens;

(10) $\forall v_0 \phi(v_1|v_1)$ by (9), and Generalisation

Hence, formulae (1) - (3) and (8) - (10) make up the required proof.