Math 3300

(Mathematical Logic)

2nd Tutorial Solutions

1. (i) $v_0[\underline{a}] = 1/2$. (ii) $C_0[\underline{a}] = 1/3$.

(iii) $F_0(v_1)[\underline{a}] = f_0(v_1[\underline{a}] = f_0(3) = -3.$

(iv) $F_1(C_0)[\underline{a}] = f_1(C_0[\underline{a}]) = f_1(1/3) = 1/(1/3) = 3.$

(v) $F_3(v_1, F_0(v_2[\underline{a}]) = f_3(v_1[\underline{a}], F_0(v_2)[\underline{a}]) = f_3(3, f_0(v_2[\underline{a}]))$
$= f_3(3, f_0(1/4)) = 3(-1/4) = -3/4.$

(vi) $F_3(F_2(F_1(v_1), C_0), F_1(v_0))[\underline{a}] = f_3(F_2(F_1(v_1), C_0)[\underline{a}], F_1(v_0)[\underline{a}])$
$= f_3(f_2(F_1(v_1)[\underline{a}], C_0[\underline{a}]), F_1(v_0[\underline{a}]))$
$= f_3(f_2(f_1(v_1[\underline{a}]), c_0), f_1(v_0[\underline{a}]))$
$= f_3(f_2(f_1(3), 1/3), f_1(1/2)) = f_3((1/3) + (1/3), 2)$
$= (2/3).3 = 4/3.$

2 (i) $v_0[\underline{a}] = 111.$ (ii) $C_0[\underline{a}] = 010.$

(iii) $f_0(v_1[\underline{a}]) = 0001.$ (iv) $f_1(010) = 10.$

(v) $f_3(v_1[\underline{a}], f_0(v_2[\underline{a}])) = f_3(001, 010) = 001.$

(vi) $f_3(f_2(f_1(v_1[\underline{a}]), c_0), f_1(v_0[\underline{a}])) = f_3(f_2(f_1(001), 010), f_1(111)$
$= f_3(f_2(01, 010), 11) = f_3(01010, 11) = 01010.$

3 (i) $Val(\phi, \underline{A}, \underline{a}) = T$ iff $v_0[\underline{a}] > v_1[\underline{a}]$ iff $3 > 7$ so $Val(\phi, \underline{A}, \underline{a}) = F$.

(ii) $Val(\phi, \underline{A}, \underline{a}) = T$ iff $v_0[\underline{a}] = v_2[\underline{a}]$ iff $3 = 1$ so $Val(\phi, \underline{A}, \underline{a}) = F$.

(iii) $Val(\phi, \underline{A}, \underline{a})$ is found from $Val(P(v_0, v_1), \underline{A}, \underline{a})$, $Val(P(v_1, v_2), \underline{A}, \underline{a})$ and $Val(P(v_0, v_2), \underline{A}, \underline{a})$ by truth tables. As in 3(i), these are F, T, T respectively. Hence $Val(\phi, \underline{A}, \underline{a})$ will be the truth value of $((F \,\&\, T) \Rightarrow T)$, that is, of $(F \Rightarrow T)$, which is T. So $Val(\phi, \underline{A}, \underline{a}) = T$.

(iv) $Val(\phi, \underline{A}, \underline{a}) = T$ iff for some $x \in \mathcal{N}$, $Val(P(v_0, v_1), \underline{A}, \underline{a}(0|x)) = T$ iff for some $x \in \mathcal{N}$, $x > 7$. This is true, so $Val(\phi, \underline{A}, \underline{a}) = T$.

(v) $Val(\phi, \underline{A}, \underline{a}) = T$ iff for all $x \in \mathcal{N}$, $Val(P(v_0, v_1), \underline{A}, \underline{a}(0|x)) = T$ iff for all $x \in \mathcal{N}$, $x > 7$. This is false, so $Val(\phi, \underline{A}, \underline{a}) = F$.

4 (i) $Val(\phi, \underline{A}, \underline{a}) = T$ iff $(v_0[\underline{a}], v_1[\underline{a}]) \in R$ iff $(a, c) \in R$. Since this is not true, $Val(\phi, \underline{A}, \underline{a}) = F$.

(ii) $Val(\phi, \underline{A}, \underline{a}) = T$ iff $v_0[\underline{a}] = v_2[\underline{a}]$ iff $a = a$. So $Val(\phi, \underline{A}, \underline{a}) = T$.

(iii) $Val(\phi, \underline{A}, \underline{a})$ is found from $Val(P(v_0, v_1), \underline{A}, \underline{a})$, $Val(P(v_1, v_2), \underline{A}, \underline{a})$ and $Val(P(v_0, v_2), \underline{A}, \underline{a})$ by truth tables. As in 4(i), $Val(P(v_0, v_1), \underline{A}, \underline{a}) = F$, $Val(P(v_1, v_2), \underline{A}, \underline{a}) = T$ since $(c, a) \in R$, and $Val(P(v_0, v_2), \underline{A}, \underline{a}) = T$ since $(a, a) \in R$. Hence $Val(\phi, \underline{A}, \underline{a})$ is the truth value of $((F \,\&\, T) \Rightarrow F)$, which is T.

(iv) $Val(\phi, \underline{A}, \underset{\sim}{a}) = T$ iff for some $x \in A$, $Val(P(v_0, v_1), \underline{A}, \underset{\sim}{a}(0|x)) = T$ iff for some $x \in A$, $(x, c) \in R$. This is true since $(c, c) \in R$, so $Val(\phi, \underline{A}, \underset{\sim}{a})=T$.

(v) $Val(\phi, \underline{A}, \underset{\sim}{a}) = T$ iff for all $x \in A$, $Val(P(v_0, v_1), \underline{A}, \underset{\sim}{a}(0|x)) = T$ iff for all $x \in A$, $(x, c) \in R$. This is false since, for example, $(a, c) \notin R$. Hence $Val(\phi, \underline{A}, \underset{\sim}{a}) = F$.

5 For each part, let $S = \{(a_1, a_2) \in \mathcal{N} \times \mathcal{N} \mid Val(\phi, \underline{A}, \underset{\sim}{a}) = T\}$.

(i) $Val(\phi, \underline{A}, \underset{\sim}{a}) = T$ iff $F_0(v_1, v_2)[\underset{\sim}{a}] \geq C_0[\underset{\sim}{a}]$ iff $a_1 \times a_2 \geq 1$ iff $a_1 \geq 1$ and $a_2 \geq 1$. Thus $S = \{(a_1, a_2) \mid a_1 \geq 1 \text{ and } a_2 \geq 1\}$.

(ii) $Val(\phi, \underline{A}, \underset{\sim}{a}) = T$ iff ($a_1 \geq a_2$ implies that $a_2 \geq a_1$) is true. When does this hold ?

(a) If $a_1 \geq a_2$ is *true*, then we must also have $a_2 \geq a_1$ true, and hence $a_1 = a_2$.

(b) If $a_1 \geq a_2$ is *false*, then the implication is true. Now $a_1 \geq a_2$ is false precisely when $a_1 < a_2$. Hence

$S = \{(a_1, a_2) \mid a_1 = a_2 \text{ or } a_1 < a_2\} = \{(a_1, a_2) \mid a_1 \leq a_2\}$.

(iii) $Val(\phi, \underline{A}, \underset{\sim}{a}) = T$ iff there is an $x \in \mathcal{N}$ such that $Val(P(C, F(v_1, v_2)), \underline{A}, \underset{\sim}{a}(2|x)) = T = Val(\exists v_1 \sim P(v_1, v_2), \underline{A}, \underset{\sim}{a}(2|x)) = T$. When does this hold ?

(a) $Val(P(C, F(v_1, v_2), \underline{A}, \underset{\sim}{a}(2|x)) = T$ iff $1 \geq a_1 \times x$, since $\underset{\sim}{a}(2|x) = \langle a_0, a_1, x, a_3, \ldots \rangle$.

(b) $Val(\exists v_1 \sim P(v_1, v_2), \underline{A}, \underset{\sim}{a}(2|x)) = T$ iff there is $y \in \mathcal{N}$ such that $Val(\sim P(v_1, v_2), \underline{A}, \underset{\sim}{a}(2|x)(1|y)) = T$ iff there is $y \in \mathcal{N}$ with $y \not\geq x$ (since $a(2|x)(1|y) = \langle a_0, y, x, a_3, \ldots \rangle$) iff there is $y \in \mathcal{N}$ with $y < x$ iff $x \geq 1$.

Hence $Val(\phi, \underline{A}, \underset{\sim}{a}) = T$ iff there is $x \in \mathcal{N}$ such that $1 \geq a_1 \times x$ and $x \geq 1$, iff $a_1 = 0$ or 1. Thus $S = \{(a_1, a_2) \mid a_1 = 0 \text{ or } a_1 = 1,\ a_2 \text{ arbitrary}\}$.

(iv) $Val(\phi, \underline{A}, \underset{\sim}{a}) = T$ iff $Val(P(C, v_1), \underline{A}, \underset{\sim}{a}) = T$ *and* $Val(\sim(v_1 \equiv C), \underline{A}, \underset{\sim}{a})=T$ *and* $Val(\exists v_0(v_2 \equiv F(v_0, v_0)), \underline{A}, \underset{\sim}{a}) = T$. When does this hold ?

(a) $Val(P(C, v_1), \underline{A}, \underset{\sim}{a}) = T$ iff $1 \geq a_1$.

(b) $Val(\sim(v_1 \equiv C), \underline{A}, \underset{\sim}{a}) = T$ iff $a_1 \neq 1$.

(c) $Val(\exists v_0(v_2 \equiv F(v_0, v_0)), \underline{A}, \underset{\sim}{a}) = T$ iff there is $x \in \mathcal{N}$ with $Val(v_2 \equiv F(v_0, v_0), \underline{A}, \underset{\sim}{a}(0|x)) = T$ iff there is $x \in \mathcal{N}$ with $a_2 = x \times x$.

Thus $Val(\phi, \underline{A}, \underset{\sim}{a}) = T$ iff $a_1 = 0$ and $a_2 = x^2$ for some $x \in \mathcal{N}$, so

$S = \{(a_1, a_2) \mid a_1 = 0 \text{ and } a_2 \text{ is a perfect square}\}$

6 In each part, let ϕ_1 be the (supposedly) valid formula and ϕ_2 the non-valid one.

(i) To show ϕ_1 is valid, take any structure $\underline{A} = \langle A, R, \ldots \rangle$ and assignmen[t] $\underset{\sim}{a}$ from A. We need to show that $Val(\phi_1, \underline{A}, \underset{\sim}{a}) = T$. Now ϕ_1 is an implicati[on]. Hence if $Val(\forall v_0\ P_0(v_0, v_0), \underline{A}, \underset{\sim}{a}) = F$, then $Val(\phi_1, \underline{A}, \underset{\sim}{a}) = T$. So we nee[d] only check when $Val(\forall v_0\ P_0(v_0, v_0), \underline{A}, \underset{\sim}{a}) = T$ to see whether $Val(\exists v_0\ P_0(v_0, v_0), \underline{A}, \underset{\sim}{a}) = T$. So we have for all $x \in A$, $R(x, x)$ is true. Then for some (any one) $x \in A$, $R(x, x)$ is true (since A is non-empty by th[e] definition of structure), and so $Val(\exists v_0\ P_0(v_0, v_0), \underline{A}, \underset{\sim}{a}) = T$. Thus for every $\underline{A}$ and $\underset{\sim}{a}$, $Val(\phi_1, \underline{A}, \underset{\sim}{a}) = T$, so ϕ_1 is valid.

To show that ϕ_2 is *not* valid, we need to find just one $\underline{A} = \langle A, R, \ldots \rangle$ and assignment $\underset{\sim}{a}$ with $Val(\phi_2, \underline{A}, \underset{\sim}{a}) = F$. For example, let $A = \{a, b\}$, let $\underset{\sim}{a}$ be any assignment from this A, and let $R = \{(a,a)\}$. Then there is $x \in A$ (namely $x = a$) with $R(x, x)$ true, so $Val(\exists v_0\ P_0(v_0, v_0), \underline{A}, \underset{\sim}{a}) = T$. But ther[e] is $x \in A$ (namely $x = b$) with $R(x,x)$ false, so $Val(\forall v_0\ P_0(v_0, v_0), \underline{A}, \underset{\sim}{a}) = F$. Hence $Val(\phi_2, \underline{A}, \underset{\sim}{a}) = F$, so ϕ_2 is not valid.

(ii) To show ϕ_1 valid, take any structure $\underline{A} = \langle A, R, \ldots \rangle$ and assignment $\underset{\sim}{a}$, as in (i) above. Again ϕ_1 is an implication, so we need check only when $Val(\exists v_1 \forall v_0\ P_0(v_0, v_1), \underline{A}, \underset{\sim}{a}) = T$, whether $Val(\forall v_0 \exists v_1\ P(v_0, v_1), \underline{A}, \underset{\sim}{a}) = T$. Since $Val(\exists v_1\ \forall v_0\ P_0(v_0, v_1), \underline{A}, \underset{\sim}{a}) = T$, there is $x_1 \in A$ for which $Val(\forall v_0\ P_0(v_0, x_1), \underline{A}, \underset{\sim}{a}) = T$. (NOTE: $\forall v_0\ P_0(v_0, x_1)$ is not a formula, but is used as convenient shorthand instead of spelling everything out as in the solution to 5(iii) above.) So there is $x_1 \in A$ for which every $y \in A$ gives $R(y, x_1)$ true. But then for every $y \in A$ there is $x \in A$ for which $R(y, x)$ is true (since $x = x_1$ is one possible choice). Thus for every $y \in A$, $Val(\exists v_1\ P_0(y, v_1), \underline{A}, \underset{\sim}{a}) = T$, and hence $Val(\forall v_0 \exists v_1\ P_0(v_0, v_1), \underline{A}, \underset{\sim}{a}) = T$, as required. Thus $Val(\phi_1, \underline{A}, \underset{\sim}{a}) = T$ for every $\underline{A}$ and $\underset{\sim}{a}$, so ϕ_1 is valid.

To show that ϕ_2 is *not* valid, we again need an example of $\underline{A}$ and $\underset{\sim}{a}$ with $Val(\phi_2, \underline{A}, \underset{\sim}{a}) = F$. For example, take $A = \{a, b\}$, $\underset{\sim}{a}$ any assignment from this A, and $R = \{(a, b), (b, a)\}$. Then $Val(\forall v_0 \exists v_1\ P_0(v_0, v_1), \underline{A}, \underset{\sim}{a}) = T$, but $Val(\exists v_1\ \forall v_0\ P(v_0, v_1), \underline{A}, \underset{\sim}{a}) = F$. Hence $Val(\phi_2, \underline{A}, \underset{\sim}{a}) = F$ and ϕ_2 is not valid.